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THE ENERGY PRODUCTION IN CONVECTIVE CORES IN STARS

BY

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1. The following tables give the functions

$$I_m(\xi) = \int_0^{\xi} [\theta(\xi')]^{m+3} \xi'^2 d\xi' \tag{1}$$

and

$$W_m(\xi) = \xi^3 [\theta(\xi)]^{m+3} / I_m(\xi) \tag{2}$$

where θ is the Lane-Emden function of index $n = 3/2$. The functions have been calculated for all integral values of the parameter m from 1 to 23 inclusive.

2. The tables of $I_m(\xi)$ will facilitate the calculation of the energy production in convective cores in stars. The structure of the convective core is described by:

Distance from center, $r = \alpha\xi$

Temperature, $T = T_c\theta$

Density, $\varrho = \varrho_c\theta^{3/2}$

where α is a scale factor, and T_c and ϱ_c the central temperature and density. The energy production inside a sphere of radius r, L_r, will be

$$L_r = \int_0^r 4\,\pi\, r^2 \varrho\varepsilon\, dr. \tag{3}$$

If the energy production is expressed by the power law

$$\varepsilon = \varepsilon_0 \varrho^{\delta} T^{\nu}, \tag{4}$$

we get

$$L_{r(=\alpha\xi)} = 4\,\pi\alpha^3\, \varepsilon_0\, \varrho_c^{1+\delta} T_c^{\nu} I_m(\xi) \tag{5}$$

with $m = \nu + 3\,(\delta - 1)/2$.

3. The function W is the homology invariant quantity corresponding to I. The tables are primarily calculated as an aid to starting calculations of the interior structure of stars where homology invariant quantities are used as the primary variables. Such a choice will be convenient whenever the energy production and the opacity can be expressed with sufficient accuracy by power expressions, such as (4) above for ε and

$$\varkappa = \varkappa_0 \varrho^{1-\alpha} T^{-3-s} \tag{6}$$

for the opacity, and the radiation pressure can be neglected. In this case we can choose the variables

$$\left.\begin{aligned} V &= -\frac{d \log P}{d \log r} \\ U &= \frac{d \log M_r}{d \log r} \\ W &= \frac{d \log L_r}{d \log r} \\ H &= \frac{V}{n+1} = -\frac{d \log T}{d \log r} \end{aligned}\right\} \tag{7}$$

where we have used the usual notation: P is the pressure, M_r is the mass within the sphere of radius r, and n is the polytropic index.

In terms of these variables the usual four differential equations governing the internal structure of a star in radiative equilibrium reduce themselves to the following three differential equations

$$\left.\begin{aligned} \frac{dU}{dV} &= \frac{U(3-V+H-U)}{V(U+H-1)} \\ \frac{dW}{dV} &= \frac{W(3-(1+\delta)V-(\nu-1-\delta)H-W)}{V(U+H-1)} \\ \frac{dH}{dV} &= \frac{H((9+s-\alpha)H-(2-\alpha)V+W-1)}{V(U+H-1)} \end{aligned}\right\} \tag{8}$$

and the quadrature

$$\log r/r_0 = \int \frac{dV}{V(U+H-1)}\,. \tag{9}$$

Inside the convective core the functions U, V, and H are independent of the opacity and energy production. U and V are tabulated in British Association Tables, Vol. II. The variable H follows immediately as $2\,V/5$. The remaining variable, W, is the function tabulated in the present publication. Contrary to what is the case of V, U and H, W depends on the energy production law. The present tables cover the ground where a law of the form (4), with $1 \leq \nu + 3\,(\delta - 1)/2 \leq 23$, is concerned.

The tables of W have been calculated only with the argument ξ. The applications to integrations of stellar structure require V to be the independent variable. Therefore an auxiliary table, which gives U and ξ as functions of V, has been provided.

4. In a recent paper by OSTERBROCK and the present author[1] it has been shown how an upper limit to the extent of the convective core in a star where (4) and (6) are valid can be derived. This limit has the following form: A convective core, extending to the point in the star where $V = V_0$, is only possible if

$$U_0 - W_0 - (1.2 + 0.4\,s + 0.6\,\alpha)\,V_0 \geq 0 \tag{10}$$

or

$$s + \frac{3}{2}\alpha \leq \frac{5\,(U_0 - W_0)}{2\,V_0} - 3. \tag{11}$$

If ε is of the form (4), the quantity on the right hand side of (11) depends only on V_0 and $\nu + 3\,(\delta - 1)/2$. It has been plotted in figure 1.

It is of considerable interest to know the behaviour of the convective core in a star where two energy sources, each with an output of the form (4), are active. There would be no difficulties in evaluating the W functions for such mixtures and then plotting the corresponding curves in the diagram. Since, however, the most important feature of the curve is its intersection with the $s + 3\,\alpha/2$ axis, we shall confine our attention to the evaluation of the position of this point. For this purpose, let us write the total energy production rate as

$$\varepsilon = \varepsilon_c\,(\gamma\theta^{m_1} + (1 - \gamma)\,\theta^{m_2})\,\theta^{3/2}. \tag{12}$$

[1] Ap. J., **117**, 306, 1953.

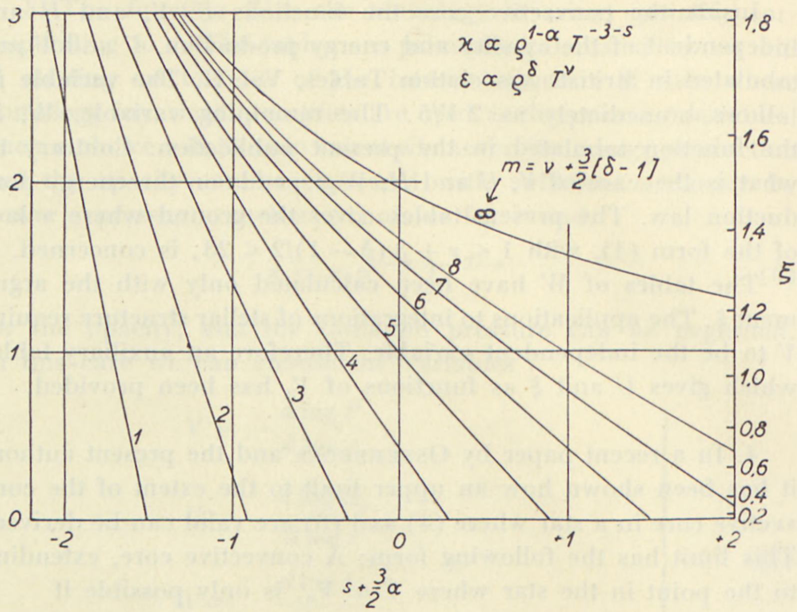

Figure 1. The curves give the upper limit to the extent of the convective core as a function of the opacity and energy production laws.

Thus ε_c is the total energy production rate at the center while γ measures the relative contribution to this total from the reaction characterized by the subscript 1. Using the power expansions of the Lane-Emden function it is easy to find the value of the relevant function at $V = 0$:

$$\left[\frac{5\,(U - W)}{2\,V}\right]_{V \to 0} - 3 = 0.6\,\gamma\, m_1 + 0.6\,(1 - \gamma)\, m_2 - 2.1. \qquad (13)$$

Thus the abscissa of the required intersection on the horizontal axis is a mean of the values of m for the two reactions, the weighing factor being the contribution from the particular process to the total energy production at the center.

The tables.

The tables giving ξ and U as functions of V have been obtained by interpolation of the tables given in British Association Tables, Vol. II. The second difference, Δ'', or the modified second diference, $M'' = \Delta'' - 0.184\,\Delta^{iv}$, has been given. Thus interpolation to the fraction n follows from Everett's formula:

$$f(x_i + n\Delta x) = (1-n)\,f(x_i) + nf(x_{i+1}) \\ + E_0''\Delta''(x_i) + E_1''\Delta''(x_{i+1})$$

or the same expression with M'' replacing Δ''. The coefficients E_0'' and E_1'' have been tabulated with argument n in Interpolation and Allied Tables, H. M. Stationary Office (reprinted from the Nautical Almanac for 1937). For small values of V the table for ξ becomes unmanageable. This difficulty is avoided if one works with the function $\sqrt{6V/5} - \xi$.

The tables of I and W are based on the values of θ given in British Association Tables, Vol. II. The calculations were carried out to seven decimals and only in the copy prepared for printing the functions were rounded to five and four decimals. For interpolation to arbitrary values of ξ the modified second differences have been given. Most of the work of calculating these functions was done by means of the IBM 602-A calculators at the IBM Watson Scientific Computing Laboratory, New York City.

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V	ξ	M″	U	Δ″
0.0	0.00000	—	3.00000	+ 30
0.1	0.34606	—	2.96414	30
0.2	0.48888	—	2.92858	30
0.3	0.59811	—1567	2.89332	29
0.4	0.68986	1065	2.85835	30
0.5	0.77040	776	2.82368	31
0.6	0.84294	601	2.78932	31
0.7	0.90936	477	2.75527	31
0.8	0.97093	396	2.72153	31
0.9	1.02850	333	2.68810	32
1.0	1.08271	291	2.65499	33
1.1	1.13401	251	2.62221	32
1.2	1.18280	224	2.58975	33
1.3	1.22935	200	2.55762	32
1.4	1.27390	180	2.52581	34
1.5	1.31665	163	2.49434	33
1.6	1.35777	149	2.46320	33
1.7	1.39740	139	2.43239	33
1.8	1.43564	128	2.40191	34
1.9	1.47260	118	2.37177	34
2.0	1.50838	111	2.34197	34
2.1	1.54305	105	2.31251	35
2.2	1.57667	97	2.28340	33
2.3	1.60932	93	2.25462	34
2.4	1.64104	87	2.22618	34
2.5	1.67189	83	2.19808	35
2.6	1.70191	80	2.17033	34
2.7	1.73113	75	2.14292	35
2.8	1.75960	71	2.11586	33
2.9	1.78736	70	2.08913	34
3.0	1.81442	66	2.06274	35
3.1	1.84082	63	2.03670	33
3.2	1.86659	61	2.01099	34
3.3	1.89175	58	1.98562	34
3.4	1.91633	57	1.96059	33
3.5	1.94034	55	1.93589	34
3.6	1.96380	— 53	1.91153	+ 33

ξ	I_1	M''	I_2	M''	I_3	M''	I_4	M''	I_5	M''
0.0	.00000	0	.00000	0	.00000	0	.00000	0	.00000	0
0.1	.00033	+199	.00033	+199	.00033	+198	.00033	+198	.00033	+198
0.2	.00262	382	.00261	377	.00260	373	.00259	369	.00258	364
0.3	.00868	535	.00861	520	.00853	504	.00845	490	.00838	475
0.4	.02002	647	.01970	612	.01940	578	.01909	546	.01880	515
0.5	.03773	710	.03682	647	.03593	589	.03507	534	.03424	482
0.6	.06245	720	.06030	625	.05825	538	.05628	459	.05438	388
0.7	.09428	681	.08995	550	.08585	436	.08199	337	.07833	250
0.8	.13283	596	.12502	434	.11776	298	.11101	185	.10474	+ 92
0.9	.17727	474	.16437	288	.15261	+141	.14188	+ 27	.13207	— 61
1.0	.22641	328	.20658	+130	.18887	— 16	.17303	—119	.15884	191
1.1	.27881	169	.25009	— 28	.22500	158	.20304	239	.18377	285
1.2	.33290	+ 9	.29335	170	.25961	273	.23072	323	.20592	339
1.3	.38709	—141	.33495	289	.29154	354	.25524	369	.22475	355
1.4	.43990	272	.37371	377	.32000	401	.27614	380	.24009	339
1.5	.49004	378	.40876	432	.34451	414	.29328	362	.25209	302
1.6	.53645	455	.43955	456	.36493	399	.30684	325	.26109	252
1.7	.57837	502	.46582	452	.38141	364	.31717	275	.26757	200
1.8	.61531	521	.48761	426	.39426	316	.32476	222	.27205	150
1.9	.64710	514	.50517	384	.40397	262	.33012	170	.27502	107
2.0	.67377	487	.51892	332	.41105	209	.33378	125	.27690	73
2.1	.69561	445	.52935	276	.41604	159	.33616	88	.27804	47
2.2	.71301	392	.53703	221	.41943	116	.33766	59	.27870	29
2.3	.72651	334	.54250	170	.42165	81	.33856	37	.27907	17
2.4	.73668	275	.54625	126	.42303	54	.33907	23	.27926	9
2.5	.74409	219	.54872	90	.42386	35	.33935	13	.27935	5
2.6	.74930	168	.55029	61	.42433	21	.33949	7	.27939	2
2.7	.75283	124	.55123	40	.42458	12	.33955	3	.27941	— 1
2.8	.75510	87	.55176	24	.42471	6	.33958	2	.27942	0
2.9	.75649	58	.55205	14	.42477	3	.33960	— 1	.27942	0
3.0	.75728	37	.55218	7	.42479	1	.33960	0		
3.1	.75770	21	.55225	3	.42480	— 1				
3.2	.75789	11	.55227	— 1	.42480	0				
3.3	.75797	5	.55228	0						
3.4	.75800	— 2	.55228	0						
3.5	.75800	0								

ξ	I_6	M''	I_7	M''	I_8	M''	I_9	M''	I_{10}	M''
0.0	.00000	0	.00000	0	.00000	0	.00000	0	.00000	0
0.1	.00033	+197	.00033	+197	.00033	+197	.00033	+196	.00033	+196
0.2	.00257	360	.00256	356	.00255	351	.00254	347	.00253	343
0.3	.00830	461	.00823	446	.00816	433	.00809	419	.00802	406
0.4	.01851	485	.01822	456	.01794	428	.01767	402	.01740	376
0.5	.03342	434	.03263	389	.03187	346	.03112	306	.03040	269
0.6	.05257	322	.05083	263	.04916	210	.04756	162	.04602	+119
0.7	.07488	174	.07161	+109	.06851	+ 52	.06558	+ 4	.06281	— 37
0.8	.09890	+ 16	.09347	— 46	.08841	— 95	.08369	—134	.07929	164
0.9	.12311	—127	.11492	175	.10741	209	.10052	232	.09420	245
1.0	.14612	237	.13468	264	.12439	278	.11512	280	.10675	276
1.1	.16682	306	.15189	309	.13869	301	.12700	285	.11662	264
1.2	.18455	333	.16607	313	.15004	286	.13609	256	.12390	226
1.3	.19901	324	.17718	287	.15858	248	.14265	210	.12894	176
1.4	.21028	290	.18546	242	.16466	198	.14712	159	.13223	126
1.5	.21867	243	.19134	190	.16877	146	.15000	111	.13425	83
1.6	.22465	190	.19530	140	.17141	102	.15175	73	.13541	51
1.7	.22872	141	.19786	97	.17301	66	.15276	44	.13604	29
1.8	.23136	99	.19942	64	.17394	40	.15330	25	.13636	16
1.9	.23300	65	.20033	39	.17444	23	.15358	14	.13652	8
2.0	.23398	41	.20083	23	.17470	12	.15372	7	.13659	4
2.1	.23452	24	.20109	12	.17483	6	.15378	3	.13662	2
2.2	.23482	14	.20122	6	.17488	3	.15380	1	.13663	— 1
2.3	.23496	7	.20128	3	.17491	1	.15381	— 1	.13663	0
2.4	.23503	4	.20131	1	.17492	— 1	.15382	0		
2.5	.23507	2	.20132	— 1	.17492	0	.15382	0		
2.6	.23508	— 1	.20132	0						
2.7	.23508	0								

ξ	I_{11}	M''	I_{12}	M''	I_{13}	M''	I_{14}	M''	I_{15}	M''
0.0	.00000	0	.00000	0	.00000	0	.00000	0	.00000	0
0.1	.00033	+196	.00033	+195	.00033	+195	.00033	+195	.00033	+194
0.2	.00252	338	.00251	334	.00250	330	.00249	326	.00248	322
0.3	.00795	393	.00788	380	.00781	368	.00774	355	.00767	343
0.4	.01713	352	.01687	328	.01662	305	.01637	283	.01612	263
0.5	.02970	234	.02901	202	.02835	172	.02771	+144	.02708	+118
0.6	.04454	+ 80	.04312	+ 45	.04176	+ 14	.04045	— 14	.03919	— 38
0.7	.06019	— 71	.05770	—100	.05534	—124	.05310	144	.05098	160
0.8	.07518	187	.07134	203	.06776	214	.06441	221	.06127	224
0.9	.08839	251	.08305	252	.07814	248	.07361	241	.06942	232
1.0	.09918	265	.09233	252	.08611	236	.08046	219	.07532	201
1.1	.10739	242	.09915	218	.09178	195	.08518	173	.07924	152
1.2	.11322	197	.10382	169	.09552	144	.08817	122	.08164	103
1.3	.11709	145	.10680	119	.09781	97	.08993	78	.08300	63
1.4	.11951	99	.10857	77	.09912	59	.09090	45	.08370	35
1.5	.12092	62	.10956	46	.09981	33	.09138	24	.08404	18
1.6	.12169	36	.11008	25	.10015	17	.09161	12	.08420	8
1.7	.12209	19	.11033	13	.10031	8	.09171	5	.08426	3
1.8	.12228	10	.11044	6	.10038	4	.09175	2	.08428	— 1
1.9	.12237	4	.11049	3	.10040	1	.09176	— 1	.08429	0
2.0	.12240	2	.11051	— 1	.10041	— 1	.09177	0	.08429	0
2.1	.12242	— 1	.11051	0	.10042	0	.09177	0		
2.2	.12242	0	.11052	0	.10042	0				
2.3			.11052	0						

ξ	I_{16}	M''	I_{17}	M''	I_{18}	M''	I_{19}	M''	I_{20}	M''
0.0	.00000	0	.00000	0	.00000	0	.00000	0	.00000	0
0.1	.00033	+194	.00033	+193	.00033	+193	.00033	+193	.00033	+192
0.2	.00247	317	.00246	313	.00245	309	.00244	305	.00243	301
0.3	.00761	332	.00754	320	.00748	309	.00741	298	.00735	287
0.4	.01588	243	.01564	223	.01541	205	.01518	188	.01496	171
0.5	.02647	+ 93	.02588	+ 71	.02530	+ 50	.02474	+ 31	.02420	+ 13
0.6	.03798	— 60	.03682	— 79	.03571	— 95	.03463	—110	.03360	—122
0.7	.04897	172	.04706	181	.04524	188	.04352	193	.04188	196
0.8	.05834	224	.05559	222	.05301	218	.05059	212	.04832	205
0.9	.06556	221	.06199	210	.05869	197	.05562	184	.05278	172
1.0	.07064	184	.06635	167	.06244	151	.05885	136	.05556	121
1.1	.07389	132	.06907	116	.06469	101	.06073	87	.05712	75
1.2	.07582	86	.07061	72	.06593	60	.06172	49	.05791	40
1.3	.07686	50	.07141	40	.06655	31	.06219	25	.05828	19
1.4	.07738	26	.07179	20	.06683	15	.06240	11	.05843	8
1.5	.07762	12	.07196	9	.06695	6	.06248	5	.05849	3
1.6	.07772	5	.07203	4	.06699	2	.06251	2	.05851	— 1
1.7	.07776	2	.07205	— 1	.06701	— 1	.06252	— 1	.05852	0
1.8	.07777	— 1	.07206	0	.06701	0	.06252	0	.05852	0
1.9	.07778	0	.07206	0						
2.0	.07778	0								

ξ	I_{21}	M''	I_{22}	M''	I_{23}	M''
0.0	.00000	0	.00000	0	.00000	0
0.1	.00033	+192	.00033	+191	.00033	+191
0.2	.00242	297	.00242	293	.00241	289
0.3	.00728	277	.00722	266	.00716	256
0.4	.01473	+155	.01452	+139	.01431	+125
0.5	.02367	— 4	.02315	— 19	.02265	— 33
0.6	.03261	133	.03165	142	.03073	149
0.7	.04033	197	.03885	197	.03744	195
0.8	.04619	198	.04418	190	.04230	181
0.9	.05014	159	.04769	147	.04541	135
1.0	.05253	108	.04975	96	.04718	85
1.1	.05383	64	.05083	55	.04808	47
1.2	.05447	33	.05134	27	.04849	22
1.3	.05475	15	.05155	12	.04865	9
1.4	.05486	6	.05164	5	.04871	3
1.5	.05490	2	.05167	— 2	.04873	— 1
1.6	.05492	— 1	.05167	0	.04874	0
1.7	.05492	0			.04874	0

ξ	W_1	M''	W_2	M''	W_3	M''	W_4	M''	W_5	M''
0.0	3.0000	—160	3.0000	—200	3.0000	—240	3.0000	—280	3.0000	—321
0.1	2.9920	160	2.9900	199	2.9880	239	2.9860	278	2.9840	318
0.2	2.9681	158	2.9601	196	2.9522	234	2.9443	272	2.9364	310
0.3	2.9284	155	2.9107	191	2.8930	227	2.8754	262	2.8578	296
0.4	2.8733	150	2.8422	184	2.8112	216	2.7804	247	2.7498	276
0.5	2.8033	144	2.7553	174	2.7078	202	2.6608	227	2.6143	250
0.6	2.7188	137	2.6510	162	2.5843	184	2.5186	203	2.4539	217
0.7	2.6206	128	2.5305	148	2.4424	163	2.3562	173	2.2719	178
0.8	2.5096	118	2.3953	131	2.2843	138	2.1766	138	2.0723	132
0.9	2.3869	106	2.2469	112	2.1124	109	1.9833	98	1.8596	79
1.0	2.2536	92	2.0875	89	1.9297	76	1.7803	53	1.6390	— 22
1.1	2.1111	77	1.9192	64	1.7396	40	1.5720	— 5	1.4163	+ 38
1.2	1.9609	60	1.7445	37	1.5454	— 1	1.3632	+ 45	1.1975	98
1.3	1.8047	41	1.5662	— 7	1.3512	+ 39	1.1589	94	0.9884	153
1.4	1.6445	— 20	1.3871	+ 23	1.1608	78	0.9639	139	0.7944	200
1.5	1.4823	+ 1	1.2104	54	0.9782	115	0.7827	177	0.6202	234
1.6	1.3203	24	1.0391	84	0.8070	148	0.6190	206	0.4691	251
1.7	1.1606	47	0.8761	112	0.6505	173	0.4756	222	0.3428	251
1.8	1.0056	69	0.7242	135	0.5112	190	0.3542	224	0.2413	236
1.9	0.8574	90	0.5857	154	0.3906	197	0.2549	214	0.1632	208
2.0	0.7183	109	0.4625	165	0.2896	193	0.1769	192	0.1057	172
2.1	0.5899	125	0.3557	169	0.2076	180	0.1179	164	0.0654	134
2.2	0.4740	136	0.2656	166	0.1435	160	0.0753	132	0.0385	98
2.3	0.3715	142	0.1920	156	0.0953	135	0.0458	100	0.0215	67
2.4	0.2832	144	0.1338	139	0.0606	108	0.0265	72	0.0113	43
2.5	0.2091	139	0.0896	120	0.0366	82	0.0145	49	0.0055	26
2.6	0.1489	130	0.0572	98	0.0209	59	0.0074	31	0.0025	15
2.7	0.1016	117	0.0346	76	0.0112	40	0.0035	18	0.0011	8
2.8	0.0659	100	0.0197	56	0.0056	25	0.0015	10	0.0004	4
2.9	0.0402	82	0.0104	39	0.0025	15	0.0006	5	0.0001	2
3.0	0.0227	64	0.0049	25	0.0010	8	0.0002	2	0.0000	+ 1
3.1	0.0116	46	0.0021	15	0.0004	4	0.0001	+ 1	0.0000	0
3.2	0.0052	31	0.0007	8	0.0001	2	0.0000	0		
3.3	0.0019	18	0.0002	3	0.0000	+ 1				
3.4	0.0005	9	0.0000	+ 1	0.0000	0				
3.5	0.0001	+ 3	0.0000	0						
3.6	0.0000	0								

ξ	W_6	M''	W_7	M''	W_8	M''	W_9	M''	W_{10}	M''
0.0	3.0000	—361	3.0000	—401	3.0000	—441	3.0000	—482	3.0000	—522
0.1	2.9820	357	2.9800	397	2.9780	436	2.9760	475	2.9741	514
0.2	2.9285	347	2.9206	384	2.9127	420	2.9048	456	2.8970	492
0.3	2.8403	329	2.8229	361	2.8056	393	2.7882	424	2.7710	454
0.4	2.7195	304	2.6893	330	2.6593	355	2.6295	377	2.6000	399
0.5	2.5683	270	2.5228	288	2.4778	304	2.4334	316	2.3894	327
0.6	2.3903	228	2.3278	236	2.2663	239	2.2058	240	2.1464	236
0.7	2.1896	178	2.1093	173	2.0310	163	1.9546	149	1.8801	130
0.8	1.9713	119	1.8737	100	1.7795	— 76	1.6886	— 46	1.6010	— 12
0.9	1.7412	— 53	1.6283	— 21	1.5206	+ 18	1.4182	+ 62	1.3209	+111
1.0	1.5059	+ 17	1.3808	+ 63	1.2635	114	1.1539	169	1.0517	227
1.1	1.2723	89	1.1396	144	1.0177	203	0.9063	262	0.8048	321
1.2	1.0475	156	0.9126	216	0.7919	275	0.6845	331	0.5894	381
1.3	0.8382	214	0.7069	272	0.5931	323	0.4951	366	0.4113	399
1.4	0.6499	256	0.5280	304	0.4262	341	0.3418	365	0.2725	376
1.5	0.4870	279	0.3791	311	0.2927	328	0.2243	331	0.1707	322
1.6	0.3515	280	0.2608	293	0.1916	290	0.1395	274	0.1008	251
1.7	0.2438	262	0.1713	255	0.1191	236	0.0820	209	0.0560	178
1.8	0.1619	228	0.1072	207	0.0702	178	0.0454	146	0.0291	116
1.9	0.1027	186	0.0637	156	0.0390	124	0.0236	94	0.0142	69
2.0	0.0621	141	0.0359	109	0.0204	80	0.0115	56	0.0064	38
2.1	0.0356	101	0.0190	71	0.0100	48	0.0052	31	0.0027	19
2.2	0.0193	67	0.0095	43	0.0046	26	0.0022	15	0.0011	9
2.3	0.0098	42	0.0044	24	0.0020	13	0.0009	7	0.0004	4
2.4	0.0047	24	0.0019	13	0.0008	6	0.0003	3	0.0001	+ 1
2.5	0.0021	13	0.0008	6	0.0003	3	0.0001	+ 1	0.0000	0
2.6	0.0008	6	0.0003	3	0.0001	+ 1	0.0000	0		
2.7	0.0003	3	0.0001	+ 1	0.0000	0				
2.8	0.0001	+ 1	0.0000	0						
2.9	0.0000	0								

ξ	W_{11}	M''	W_{12}	M''	W_{13}	M''	W_{14}	M''	W_{15}	M''
0.0	3.0000	—562	3.0000	—602	3.0000	—643	3.0000	—683	3.0000	—723
0.1	2.9721	553	2.9701	592	2.9681	631	2.9661	670	2.9641	709
0.2	2.8891	527	2.8813	562	2.8735	597	2.8657	631	2.8579	665
0.3	2.7538	483	2.7367	511	2.7196	538	2.7027	564	2.6857	589
0.4	2.5706	419	2.5415	437	2.5125	454	2.4838	469	2.4552	483
0.5	2.3459	334	2.3029	340	2.2604	343	2.2185	343	2.1770	341
0.6	2.0881	229	2.0308	219	1.9746	205	1.9194	188	1.8653	—168
0.7	1.8076	—107	1.7371	— 79	1.6685	— 48	1.6019	— 12	1.5371	+ 27
0.8	1.5167	+ 28	1.4357	+ 72	1.3579	+119	1.2832	+170	1.2116	223
0.9	1.2287	164	1.1414	219	1.0589	277	0.9812	335	0.9080	394
1.0	0.9567	286	0.8686	345	0.7871	402	0.7119	457	0.6428	509
1.1	0.7127	377	0.6295	429	0.5546	475	0.4873	515	0.4271	548
1.2	0.5057	424	0.4323	459	0.3684	484	0.3129	501	0.2649	509
1.3	0.3402	422	0.2801	434	0.2296	436	0.1875	429	0.1526	415
1.4	0.2160	377	0.1704	367	0.1338	349	0.1045	326	0.0813	298
1.5	0.1291	304	0.0971	279	0.0726	250	0.0540	220	0.0400	190
1.6	0.0724	222	0.0516	192	0.0366	161	0.0258	133	0.0181	108
1.7	0.0379	148	0.0255	119	0.0171	94	0.0114	72	0.0075	55
1.8	0.0186	90	0.0117	67	0.0074	49	0.0046	36	0.0029	25
1.9	0.0084	50	0.0050	34	0.0029	23	0.0017	16	0.0010	10
2.0	0.0036	25	0.0020	16	0.0011	10	0.0006	6	0.0003	4
2.1	0.0014	12	0.0007	7	0.0004	4	0.0002	2	0.0001	+ 1
2.2	0.0005	5	0.0002	3	0.0001	+ 1	0.0001	+ 1	0.0000	0
2.3	0.0002	2	0.0001	+ 1	0.0000	0	0.0000	0		
2.4	0.0000	+ 1	0.0000	0						
2.5	0.0000	0								

ξ	W_{16}	M''	W_{17}	M''	W_{18}	M''	W_{19}	M''	W_{20}	M''
0.0	3.0000	—764	3.0000	—804	3.0000	—845	3.0000	—885	3.0000	—926
0.1	2.9621	748	2.9601	786	2.9581	825	2.9561	864	2.9542	902
0.2	2.8501	698	2.8423	731	2.8345	764	2.8268	796	2.8190	828
0.3	2.6689	614	2.6521	638	2.6353	661	2.6187	683	2.6021	704
0.4	2.4269	495	2.3987	505	2.3708	514	2.3431	522	2.3156	528
0.5	2.1361	336	2.0956	329	2.0557	320	2.0162	308	1.9773	294
0.6	1.8122	—144	1.7602	—117	1.7092	— 88	1.6593	— 55	1.6104	— 20
0.7	1.4743	+ 69	1.4133	+114	1.3542	+162	1.2970	+212	1.2416	+265
0.8	1.1432	279	1.0777	336	1.0151	394	0.9555	452	0.8986	511
0.9	0.8392	452	0.7747	509	0.7143	564	0.6578	616	0.6051	665
1.0	0.5793	557	0.5212	599	0.4681	636	0.4198	667	0.3758	692
1.1	0.3735	573	0.3259	591	0.2837	602	0.2465	605	0.2137	602
1.2	0.2236	509	0.1883	501	0.1581	488	0.1324	469	0.1106	445
1.3	0.1237	394	0.1000	370	0.0806	342	0.0648	313	0.0519	283
1.4	0.0630	268	0.0487	238	0.0375	209	0.0288	181	0.0220	155
1.5	0.0295	162	0.0217	135	0.0159	112	0.0116	91	0.0084	74
1.6	0.0126	86	0.0088	68	0.0061	53	0.0042	41	0.0029	31
1.7	0.0049	41	0.0032	30	0.0021	22	0.0014	16	0.0009	11
1.8	0.0018	17	0.0011	12	0.0007	8	0.0004	5	0.0002	4
1.9	0.0006	7	0.0003	4	0.0002	3	0.0001	+ 2	0.0001	+ 1
2.0	0.0002	2	0.0001	+ 1	0.0000	+ 1	0.0000	0	0.0000	0
2.1	0.0000	+ 1	0.0000	0	0.0000	0				
2.2	0.0000	0								

ξ	W_{21}	M''	W_{22}	M''	W_{23}	M''
0.0	3.0000	—966	3.0000	—1007	3.0000	—1048
0.1	2.9522	940	2.9502	978	2.9482	1017
0.2	2.8113	859	2.8036	891	2.7959	921
0.3	2.5855	724	2.5690	743	2.5526	761
0.4	2.2883	532	2.2612	535	2.2343	536
0.5	1.9389	—277	1.9010	— 258	1.8636	— 237
0.6	1.5625	+ 18	1.5157	+ 59	1.4699	+ 102
0.7	1.1880	319	1.1362	375	1.0861	432
0.8	0.8445	569	0.7930	626	0.7441	682
0.9	0.5559	710	0.5102	750	0.4677	786
1.0	0.3359	710	0.2998	723	0.2672	730
1.1	0.1849	593	0.1597	579	0.1377	561
1.2	0.0922	419	0.0767	391	0.0637	362
1.3	0.0415	253	0.0331	224	0.0263	197
1.4	0.0168	131	0.0128	110	0.0097	92
1.5	0.0061	59	0.0044	47	0.0032	37
1.6	0.0020	23	0.0014	17	0.0009	13
1.7	0.0006	8	0.0004	5	0.0002	4
1.8	0.0002	2	0.0001	+ 2	0.0001	+ 1
1.9	0.0000	+ 1	0.0000	0	0.0000	0
2.0	0.0000	0				